Determinante n-tog reda

Zadatak 1. Izračunajte:

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & \dots & 0 \end{vmatrix}.$$

Rješenje.

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & \dots & 0 \end{vmatrix} = \substack{\text{prvi redak dodamo} \\ \text{svim ostalima}} = \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 2 & 6 & \dots & 2n \\ 0 & 0 & 3 & \dots & 2n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n \end{vmatrix} = 1 \cdot 2 \cdot 3 \cdots n = n!.$$

□

Zadatak 2. Izračunajte:

$$D_n = \begin{vmatrix} 3 & 2 & 2 & \dots & 2 \\ 2 & 3 & 2 & \dots & 2 \\ 2 & 2 & 3 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & 3 \end{vmatrix}.$$

Rješenje.

$$D_n = \begin{vmatrix} 3 & 2 & 2 & \dots & 2 \\ 2 & 3 & 2 & \dots & 2 \\ 2 & 2 & 3 & \dots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \dots & 3 \end{vmatrix} = \substack{\text{prvi redak oduzmemo} \\ \text{od ostalih}} = \begin{vmatrix} 3 & 2 & 2 & \dots & 2 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{vmatrix} = \substack{\text{prvom stupcu dodamo} \\ \text{preostale stupce}}$$

$$= \begin{vmatrix} 3+2(n-1) & 2 & 2 & \dots & 2 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} = 3 + 2(n-1) = 2n+1.$$

□

Zadatak 3. Izračunajte determinantu matrice $A = (a_{ij}) \in M_n(\mathbb{R})$ ako je $a_{ij} = \min\{i, j\}$, $i, j = 1, \dots, n$.

Rješenje.

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & 3 & \dots & 3 \\ 1 & 2 & 3 & 4 & \dots & 4 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & 4 & \dots & n \end{vmatrix} = {\scriptstyle \text{prvi redak pomnožen s } -k \atop \text{dodajemo } k\text{-tom retku}} = \begin{vmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ -1 & 0 & 0 & 0 & \dots & 0 \\ -2 & -1 & 0 & 0 & \dots & 0 \\ -3 & -2 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-n & 2-n & 3-n & 4-n & \dots & 0 \end{vmatrix}$$

$$= {\scriptstyle \text{razvijemo po} \atop \text{zadnjem stupcu}} = (-1)^{n+1} \begin{vmatrix} -1 & 0 & 0 & \dots & 0 \\ -2 & -1 & 0 & \dots & 0 \\ -3 & -2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-n & 2-n & 3-n & \dots & -1 \end{vmatrix} = (-1)^{n+1}(-1)^{n-1} = 1.$$

□

Zadatak 4. Izračunajte:

$$D_{n+1} = \begin{vmatrix} 1 & a & a^2 & \dots & a^n \\ b & 1 & a & \dots & a^{n-1} \\ b^2 & b & 1 & \dots & a^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^n & b^{n-1} & b^{n-2} & \dots & 1 \end{vmatrix}.$$

Rješenje.

$$D_{n+1} = {\scriptstyle \text{drugi redak pomnožen s } -a \atop \text{dodamo prvom retku}} = \begin{vmatrix} 1-ab & 0 & 0 & \dots & 0 \\ b & 1 & a & \dots & a^{n-1} \\ b^2 & b & 1 & \dots & a^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^n & b^{n-1} & b^{n-2} & \dots & 1 \end{vmatrix}$$

$$= {\scriptstyle \text{treći redak pomnožen s } -a \atop \text{dodamo drugom retku}} = \begin{vmatrix} 1-ab & 0 & 0 & \dots & 0 \\ b-b^2a & 1-ab & 0 & \dots & 0 \\ b^2 & b & 1 & \dots & a^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^n & b^{n-1} & b^{n-2} & \dots & 1 \end{vmatrix}$$

$$= {\scriptstyle \text{postupak} \atop \text{ponavljamo}} = \begin{vmatrix} 1-ab & 0 & 0 & \dots & 0 \\ b-b^2a & 1-ab & 0 & \dots & 0 \\ b^2-b^3a & b-b^2a & 1-ab & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^n & b^{n-1} & b^{n-2} & \dots & 1 \end{vmatrix} = (1-ab)^n.$$

□

Zadatak 5. Izračunajte:

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 2 & 1 & 1 & \dots & 1 & 1 & 1 \\ 0 & 2 & 1 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 2 & 1 \end{vmatrix},$$

Rješenje. Determinantu razvijemo po prvom stupcu i dobijemo

$$D_n = D_{n-1} - 2D_{n-1} = -D_{n-1}$$

pa rekurzivno nastavljamo dalje

$$D_n = -D_{n-1} = (-1)^2 D_{n-2} = \cdots = (-1)^{n-1} D_1 = (-1)^{n-1},$$

gdje je D_1 determinanta 1×1 matrice (1). $\square$

Zadatak 6. Izračunajte:

$$D_n = \begin{vmatrix} n & -1 & 0 & 0 & \dots & 0 & 0 \\ n-1 & 1 & -1 & 0 & \dots & 0 & 0 \\ n-2 & 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 0 & 0 & \dots & 1 & -1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

Rješenje.

$$\begin{aligned} D_n &\underset{\substack{\text{razvoj po} \\ \text{prvom retku}}}{=} n \begin{vmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix} + \begin{vmatrix} n-1 & -1 & 0 & \dots & 0 & 0 \\ n-2 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 0 & \dots & 1 & -1 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{vmatrix} \\ &= n + D_{n-1} \\ &\underset{\substack{\text{rekurzivno} \\ \text{dalje}}}{=} \\ &= n + (n-1) + D_{n-2} \\ &= \cdots \\ &= n + (n-1) + (n-2) + \ldots + 3 + D_2 \\ &= n + (n-1) + (n-2) + \ldots + 3 + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \\ &= n + (n-1) + (n-2) + \ldots + 3 + 2 + 1 \\ &= \frac{n(n+1)}{2}. \end{aligned}$$

$\square$