Linearna algebra 1

Probni 2. kolokvij

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1. Za $n \in \mathbb{N}$ izračunajte determinantu

$$D_n = \begin{vmatrix} 1 & n & n & \dots & n \\ n & 2 & n & \dots & n \\ n & n & 3 & \dots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \dots & n \end{vmatrix}.$$

Rješenje.

$$\begin{vmatrix} 1 & n & n & \dots & n & n \\ n & 2 & n & \dots & n & n \\ n & n & 3 & \dots & n & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & n & n & \dots & n-1 & n \\ n & n & n & \dots & n & n \end{vmatrix} = \substack{\text{prvi redak oduzmemo} \\ \text{od svih ostalih}} = \begin{vmatrix} 1 & n & n & \dots & n & n \\ n-1 & 2-n & 0 & \dots & 0 & 0 \\ n-1 & 0 & 3-n & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n-1 & 0 & 0 & \dots & -1 & 0 \\ n-1 & 0 & 0 & \dots & 0 & 0 \end{vmatrix}$$

$$= \text{razvoj po zadnjem stupcu}$$

$$= (-1)^{n+1} n \begin{vmatrix} n-1 & 2-n & 0 & \dots & 0 \\ n-1 & 0 & 3-n & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n-1 & 0 & 0 & \dots & -1 \\ n-1 & 0 & 0 & \dots & 0 \end{vmatrix}$$

$$= \text{razvoj po zadnjem retku}$$

$$= (-1)^{(n-1)+1}(n-1)(-1)^{n+1} n \begin{vmatrix} 2-n & 0 & \dots & 0 \\ 0 & 3-n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{vmatrix}$$

$$= -n(n-1)(2-n)(3-n)\cdots(-2)(-1)$$

$$= \text{izlučimo } -1 \text{ iz zadnjih } n-2 \text{ faktora}$$

$$= -(-1)^{n-2} n(n-1)(n-2)(n-3)\cdots 2 \cdot 1$$

$$= (-1)^{n-1} n!.$$

2. Za koje $\lambda \in \mathbb{R}$ je matrica

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 0 & 0 & \lambda \\ \lambda & 2 & 1 & 2 \\ 4 & 0 & 0 & 1 \end{pmatrix}$$

regularna? Za sve takve λ nađite joj inverz A^{-1}.

Rješenje. Pokušajmo naći inverz matrice A elementarnim transformacijama redaka. Imamo

$$\left(\begin{array}{cccc|cccc} \textcircled{1} & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & \lambda & 0 & 1 & 0 & 0 \\ \lambda & 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & \textcircled{-2} & 0 & \lambda-2 & -2 & 1 & 0 & 0 \\ 0 & 2-\lambda & 1 & 2-\lambda & -\lambda & 0 & 1 & 0 \\ 0 & -4 & 1 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 1-\frac{\lambda}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 2-\lambda & 1 & 2-\lambda & -\lambda & 0 & 1 & 0 \\ 0 & -4 & 1 & -3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 3 & 6 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1-\frac{\lambda}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & \lambda-6 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{\lambda}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 1-\frac{\lambda}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \lambda-\frac{1}{2}\lambda^2 & -2 & 1-\frac{\lambda}{2} & 1 & 0 \\ 0 & 0 & 0 & 1-2\lambda & 0 & -2 & 0 & 1 \end{array}\right).$$

Ako je $1-2\lambda=0$, tada je zadnji redak nulredak pa je matrica A ekvivalentna matrici ranga manjeg od 4. Slijedi da je tada A singularna. U nastavku stoga pretpostavljamo da je $1-2\lambda \neq 0$ i dijelimo zadnji redak tim brojem.

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{\lambda}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 1-\frac{\lambda}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \lambda-\frac{1}{2}\lambda^2 & -2 & 1-\frac{\lambda}{2} & 1 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 & \frac{-2}{1-2\lambda} & 0 & \frac{1}{1-2\lambda} \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{\frac{1}{2}}{1-2\lambda} & 0 & \frac{-\lambda}{1-2\lambda} \\ 0 & 1 & 0 & 0 & 1 & \frac{\frac{3}{2}}{1-2\lambda} & 0 & \frac{1-\frac{\lambda}{2}}{1-2\lambda} \\ 0 & 0 & 1 & 0 & -2 & \frac{\frac{\lambda}{2}-1}{1-2\lambda} & 1 & \frac{\frac{\lambda^2}{2}-\lambda}{1-2\lambda} \\ 0 & 0 & 0 & 1 & 0 & \frac{-2}{1-2\lambda} & 0 & \frac{1}{1-2\lambda} \end{array}\right).$$

Vidimo da inverz postoji za sve $\lambda \in \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$ i dan je s

$$A^{-1} = \begin{pmatrix} 0 & \frac{\frac{1}{2}}{1-2\lambda} & 0 & \frac{-\lambda}{1-2\lambda} \\ 1 & \frac{\frac{3}{2}}{1-2\lambda} & 0 & \frac{1-\frac{\lambda}{2}}{1-2\lambda} \\ -2 & \frac{\frac{\lambda}{2}-1}{1-2\lambda} & 1 & \frac{\frac{\lambda^2}{2}-\lambda}{1-2\lambda} \\ 0 & \frac{-2}{1-2\lambda} & 0 & \frac{1}{1-2\lambda} \end{pmatrix}.$$

3. Zadane su regularne matrice $A, B \in M_4(\mathbb{R})$ kao

$$A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ -1 & 3 & 1 & 2 \\ -3 & 0 & 3 & 5 \\ 4 & 2 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 1 & 2 \\ 2 & 1 & 3 & 3 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Izračunajte $\det(A^t B^{-1} A B^2)$.

Rješenje. Primijetimo da je

$$BB^{-1} = I \implies \det(B)\det(B^{-1}) = \det I = 1 \implies \det(B^{-1}) = (\det B)^{-1}.$$

Prema Binet-Cauchyevom teoremu imamo

$$\begin{aligned}\det(A^t B^{-1} A B^2) &= \det(A^t)\det(B^{-1})\det(A)\det(B^2)\\ &= \det(A)\det(B)^{-1}\det(A)\det(B)^2\\ &= \det(A)^2\det(B)\end{aligned}$$

pa je dovoljno izračunati $\det A$ i $\det B$. Imamo

$$\begin{aligned}\det A &= \begin{vmatrix} 2 & \textcircled{1} & 0 & -1\\ -1 & 3 & 1 & 2\\ -3 & 0 & 3 & 5\\ 4 & 2 & 1 & 0\end{vmatrix}\\ &= \begin{vmatrix} 2 & 1 & 0 & -1\\ -7 & 0 & 1 & 5\\ -3 & 0 & 3 & 5\\ 0 & 0 & 1 & 2\end{vmatrix}\\ &= \text{razvijemo po drugom stupcu}\\ &= -\begin{vmatrix} -7 & 1 & 5\\ -3 & 3 & 5\\ 0 & 1 & 2\end{vmatrix}\\ &= \text{koristimo Sarrusovo pravilo}\\ &= -(-42-15+35+6)\\ &= 16.\end{aligned}$$

S druge strane, imamo

$$\begin{aligned}\det B &= \begin{vmatrix} \textcircled{1} & 2 & 3 & 4\\ 3 & 3 & 1 & 2\\ 2 & 1 & 3 & 3\\ 4 & 3 & 2 & 1\end{vmatrix}\\ &= \begin{vmatrix} 1 & 2 & 3 & 4\\ 0 & -3 & -8 & -10\\ 0 & -3 & -3 & -5\\ 0 & -5 & -10 & -15\end{vmatrix}\\ &= \text{razvijemo po prvom stupcu}\\ &= \begin{vmatrix} 3 & -2 & 0\\ -3 & -3 & \textcircled{-5}\\ 4 & -1 & 0\end{vmatrix}\\ &= \text{razvijemo po trećem stupcu}\\ &= 5\begin{vmatrix} 3 & -2\\ 4 & -1\end{vmatrix}\\ &= 25.\end{aligned}$$

Zaključujemo $\det(A^t B^{-1} A B^2) = \det(A)^2 \det(B) = 16^2 \cdot 25 = 6400$.

4. Riješite sustav u ovisnosti o parametru $\lambda \in \mathbb{R}$:

$$\begin{cases} \lambda x_1 + \ \ x_2 + \ \ x_3 + \ \ x_4 = 1, \\ \ \ x_1 + \lambda x_2 + \ \ x_3 + \ \ x_4 = 1, \\ \ \ x_1 + \ \ x_2 + \lambda x_3 + \ \ x_4 = 1, \\ \ \ x_1 + \ \ x_2 + \ \ x_3 + \lambda x_4 = 1. \end{cases}$$

Rješenje. Kad sustav zapišemo matrično, radimo elementarne transformacije redaka:

$$\left(\begin{array}{cccc|c} \lambda & 1 & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 & 1 \\ 1 & 1 & \lambda & 1 & 1 \\ 1 & 1 & 1 & \lambda & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} \textcircled{1} & \lambda & 1 & 1 & 1 \\ \lambda & 1 & 1 & 1 & 1 \\ 1 & 1 & \lambda & 1 & 1 \\ 1 & 1 & 1 & \lambda & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & \lambda & 1 & 1 & 1 \\ 0 & 1-\lambda^2 & 1-\lambda & 1-\lambda & 1-\lambda \\ 0 & 1-\lambda & \lambda-1 & 0 & 0 \\ 0 & 1-\lambda & 0 & \lambda-1 & 0 \end{array}\right).$$

Ako je $\lambda = 1$, tada ova matrica glasi:

$$\left(\begin{array}{cccc|c} 1 & \lambda & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pa očitavamo rješenje

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \alpha, \beta, \gamma \in \mathbb{R}.$$

Nastavljamo dalje uz pretpostavku $\lambda \neq 1$. Tada zadnja tri retka možemo podijeliti s $1 - \lambda$ pa imamo

$$\sim \left(\begin{array}{cccc|c} 1 & \lambda & 1 & 1 & 1 \\ 0 & 1+\lambda & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & \lambda & 1 & 1 & 1 \\ 0 & \textcircled{1} & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1+\lambda & 1 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & \lambda+1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & \textcircled{-1} & 1 & 0 \\ 0 & 0 & 1 & \lambda+2 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & \lambda+2 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & \lambda+3 & 1 \end{array}\right).$$

Ako je $\lambda = -3$, tada jednadžba koju očitavamo iz zadnjeg retka glasi $0 = 1$ pa sustav nema rješenja. Nastavljamo dalje uz pretpostavku $\lambda \notin \{1, -3\}$. Dijeljenjem zadnjeg retka s $\lambda + 3$ dobivamo

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & \lambda+2 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \textcircled{1} & \frac{1}{\lambda+3} \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{\lambda+3} \\ 0 & 1 & 0 & 0 & \frac{1}{\lambda+3} \\ 0 & 0 & 1 & 0 & \frac{1}{\lambda+3} \\ 0 & 0 & 0 & 1 & \frac{1}{\lambda+3} \end{array}\right).$$

Vidimo da sustav ima jedinstveno rješenje

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda+3} \\ \frac{1}{\lambda+3} \\ \frac{1}{\lambda+3} \\ \frac{1}{\lambda+3} \end{pmatrix}.$$