Linearna algebra 1

2. kolokvij

30. 01. 2024.

1. Za $n \geq 2$ izračunajte determinantu

$$D_n = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 3 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 3 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 3 & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 3 & 1 \\ 2 & 1 & 1 & \dots & 1 & 1 & 3 \end{vmatrix}.$$

Rješenje.

$$\begin{aligned}
&\begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 3 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 3 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 3 & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 3 & 1 \\ 2 & 1 & 1 & \dots & 1 & 1 & 3 \end{vmatrix} = \substack{\text{prvim retkom poništimo} \\ \text{ostale elemente u prvom stupcu}} \\
&= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 0 & 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 2 & 0 \\ 0 & -1 & -1 & \dots & -1 & -1 & 1 \end{vmatrix} \\
&= \text{razvoj po prvom stupcu} \\
&= \begin{vmatrix} 2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & 0 & 0 \\ 0 & 0 & \dots & 0 & 2 & 0 \\ -1 & -1 & \dots & -1 & -1 & 1 \end{vmatrix} \\
&= \text{donjetrokutasta matrica} \\
&= \underbrace{2 \cdot 2 \cdots 2}_{n-2 \text{ puta}} \cdot 1 \\
&= 2^{n-2}.
\end{aligned}$$

2. Za koje $\lambda \in \mathbb{R}$ je matrica

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 3 & 2 & 3 & 8 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & \lambda \end{pmatrix}$$

regularna? Za sve takve λ nađite joj inverz A^{-1}.

Rješenje. Pokušajmo naći inverz matrice A elementarnim transformacijama redaka. Imamo

$$\left(\begin{array}{cccc|cccc} \textcircled{1} & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 3 & 2 & 3 & 8 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & \lambda & 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & \textcircled{-1} & 3 & 5 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & \lambda-1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 3 & 6 & -2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 5 & -3 & 1 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & \lambda-6 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & -3 & 0 \\ 0 & -1 & 0 & -1 & -3 & 1 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda-2 & 2 & -1 & 2 & 1 \end{array}\right).$$

Ako je $\lambda - 2 = 0$, tada je zadnji redak nulredak pa je matrica A ekvivalentna matrici ranga manjeg od 4. Slijedi da je tada A singularna. U nastavku stoga pretpostavljamo da je $\lambda - 2 \neq 0$ i dijelimo zadnji redak tim brojem.

$$\sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & -3 & 0 \\ 0 & 1 & 0 & 1 & 3 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \textcircled{1} & \frac{2}{\lambda-2} & \frac{-1}{\lambda-2} & \frac{2}{\lambda-2} & \frac{1}{\lambda-2} \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & \frac{3\lambda-8}{\lambda-2} & \frac{3-\lambda}{\lambda-2} & \frac{3\lambda-8}{\lambda-2} & \frac{-1}{\lambda-2} \\ 0 & 0 & 1 & 0 & \frac{-4}{\lambda-2} & \frac{2}{\lambda-2} & \frac{\lambda-6}{\lambda-2} & \frac{-2}{\lambda-2} \\ 0 & 0 & 0 & 1 & \frac{2}{\lambda-2} & \frac{-1}{\lambda-2} & \frac{2}{\lambda-2} & \frac{1}{\lambda-2} \end{array}\right).$$

Vidimo da inverz postoji za sve $\lambda \in \mathbb{R} \setminus \{2\}$ i dan je s

$$A^{-1} = \begin{pmatrix} -2 & 1 & -3 & 0 \\ \frac{3\lambda-8}{\lambda-2} & \frac{3-\lambda}{\lambda-2} & \frac{3\lambda-8}{\lambda-2} & \frac{-1}{\lambda-2} \\ \frac{-4}{\lambda-2} & \frac{2}{\lambda-2} & \frac{\lambda-6}{\lambda-2} & \frac{-2}{\lambda-2} \\ \frac{2}{\lambda-2} & \frac{-1}{\lambda-2} & \frac{2}{\lambda-2} & \frac{1}{\lambda-2} \end{pmatrix}.$$

3. Zadane su regularne matrice $A, B \in M_6(\mathbb{R})$ kao

$$A = \begin{pmatrix} 1 & -1 & 0 & -1 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 8 & 7 & 8 & 7 & -8 \\ -8 & 0 & -7 & 0 & 0 & 8 \\ 0 & 0 & 8 & -7 & 8 & 8 \\ -1 & 8 & 0 & 7 & 7 & 7 \\ 8 & -8 & 7 & -7 & 7 & 0 \\ 0 & 7 & 7 & 0 & 8 & 8 \end{pmatrix}.$$

Izračunajte $\det(A^2 B A^t B^{-1})$.

Rješenje. Prema Binet-Cauchyevom teoremu imamo

$$\begin{aligned}\det(A^2BA^tB^{-1}) &= \det(A)^2\det(B)\det(A^t)\det(B^{-1})\\ &= \det(A)^2\det(B)\det(A)\det(B)^{-1}\\ &= \det(A)^3\end{aligned}$$

pa je dovoljno izračunati $\det A$. Imamo

$$\begin{aligned}\begin{vmatrix} \textcircled{1} & -1 & 0 & -1 & 1 & -1\\ -1 & 1 & -1 & 0 & 1 & 1\\ 0 & -1 & 1 & -1 & 1 & -1\\ -1 & 0 & -1 & 1 & 1 & 1\\ 1 & -1 & 1 & -1 & 1 & 0\\ 0 & 1 & 1 & 0 & 0 & 1\end{vmatrix} &= \begin{vmatrix} 1 & -1 & 0 & -1 & 1 & -1\\ 0 & 0 & -1 & -1 & 2 & 0\\ 0 & -1 & 1 & -1 & 1 & -1\\ 0 & -1 & -1 & 0 & 2 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0 & 0 & 1\end{vmatrix}\\
&= \text{razvijemo po prvom stupcu}\\
&= \begin{vmatrix} 0 & -1 & -1 & 2 & 0\\ -1 & 1 & -1 & 1 & -1\\ -1 & -1 & 0 & 2 & 0\\ 0 & 1 & 0 & 0 & \textcircled{1}\\ 1 & 1 & 0 & 0 & 1\end{vmatrix}\\
&= \begin{vmatrix} 0 & -1 & -1 & 2 & 0\\ -1 & 2 & -1 & 1 & 0\\ -1 & -1 & 0 & 2 & 0\\ 0 & 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 0\end{vmatrix}\\
&= \text{razvijemo po zadnjem retku}\\
&= \begin{vmatrix} -1 & -1 & 2 & 0\\ 2 & -1 & 1 & 0\\ -1 & 0 & 2 & 0\\ 1 & 0 & 0 & 1\end{vmatrix}\\
&= \text{razvijemo po zadnjem stupcu}\\
&= \begin{vmatrix} -1 & -1 & 2\\ 2 & -1 & 1\\ -1 & 0 & 2\end{vmatrix}\\
&= \text{koristimo Sarrusovo pravilo}\\
&= 2+1-2+4\\
&= 5.\end{aligned}$$

Zaključujemo $\det(A^2BA^tB^{-1}) = \det(A)^3 = 125$.

4. Riješite sustav u ovisnosti o parametru $\lambda \in \mathbb{R}$:

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 - x_4 = 5,\\ 4x_1 + 5x_2 + 6x_3 - 2x_4 = 7,\\ 6x_1 + 7x_2 + 8x_3 - 3x_4 = 9,\\ \lambda x_1 + 9x_2 + 10x_3 - 4x_4 = 11.\end{cases}$$

Rješenje. Kad sustav zapišemo matrično, radimo elementarne transformacije redaka:

$$\left(\begin{array}{cccc|c} \textcircled{2} & 3 & 4 & -1 & 5 \\ 4 & 5 & 6 & -2 & 7 \\ 6 & 7 & 8 & -3 & 9 \\ \lambda & 9 & 10 & -4 & 11 \end{array}\right) \sim \left(\begin{array}{cccc|c} 2 & 3 & 4 & -1 & 5 \\ 0 & \textcircled{1} & 2 & 0 & 3 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 9-\frac{3\lambda}{2} & 10-2\lambda & \frac{\lambda}{2}-4 & 11-\frac{5\lambda}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 2 & 0 & -2 & -1 & -4 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda-8 & \frac{\lambda}{2}-4 & 2\lambda-16 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & -2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & \lambda-8 & \frac{\lambda}{2}-4 & 2\lambda-16 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Ako je $\lambda = 8$, tada ova matrica glasi:

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & -2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

pa očitavamo rješenje

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{R}.$$

Nastavljamo dalje uz pretpostavku $\lambda \neq 8$. Tada treći redak možemo podijeliti s $\lambda - 8$ pa imamo

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & -1 & -\frac{1}{2} & -2 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & \textcircled{1} & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Odavde očitavamo rješenje

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}, \qquad \alpha \in \mathbb{R}.$$

Linearna algebra 1

2. domaća zadaća

16. 02. 2024.

1. Neka je $A \in M_n(\mathbb{F})$ matrica koja zadovoljava $A^2 = I$. Dokažite da je matrica $A + I$ regularna ako i samo ako je $A = I$.

Rješenje. Ako je $A = I$, tada je $A + I = 2I$, što je očito regularna matrica s inverzom $(2I)^{-1} = \frac{1}{2}I$.

Obratno, pretpostavimo da je $A + I$ regularna matrica. Primijetimo da vrijedi

$$0 = A^2 - I = (A + I)(A - I).$$

Množenjem obje strane ove jednakosti slijeva s $(A + I)^{-1}$ dobivamo $0 = A - I$ odnosno $A = I$.

2. Neka je $A \in M_n(\mathbb{F})$ matrica koja zadovoljava $A^3 = 0$. Dokažite da je tada matrica $(A - I)(A - 2I)$ regularna. *Uputa: inverz možete potražiti u obliku $\alpha I + \beta A + \gamma A^2$ za neke koeficijente $\alpha, \beta, \gamma \in \mathbb{F}$.*

Rješenje. Imamo

$$\begin{aligned}(\alpha I + \beta A + \gamma A^2)(A - I)(A - 2I) &= (\alpha I + \beta A + \gamma A^2)(A^2 - 3A + 2I) \\ &= (2\alpha)I + (-3\alpha + 2\beta)A + (\alpha - 3\beta + 2\gamma)A^2.\end{aligned}$$

Ovaj izraz je jednak I ako i samo ako je

$$\begin{cases} 2\alpha = 1, \\ 3\alpha + 2\beta = 0, \\ \alpha - 3\beta + 2\gamma = 0 \end{cases}$$

odakle lako dobivamo jedinstveno rješenje $\alpha = \frac{1}{2}, \beta = \frac{3}{4}, \gamma = \frac{7}{8}$. Dakle, kandidat za inverz je matrica

$$\frac{1}{2}I + \frac{3}{4}A + \frac{7}{8}A^2$$

za koju se lako provjeri i druga jednakost

$$(A - I)(A - 2I)\left(\frac{1}{2}I + \frac{3}{4}A + \frac{7}{8}A^2\right) = I$$

pa zaključujemo da je matrica $(A-I)(A-2I)$ regularna s inverzom $A^{-1} = \frac{1}{2}I + \frac{3}{4}A + \frac{7}{8}A^2$.

3. Dokažite da za sve matrice

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in M_2(\mathbb{R})$$

vrijedi formula

$$\begin{vmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{21} \\ c_{11} & c_{12} & 1 & 0 \\ c_{21} & c_{22} & 0 & 1 \end{vmatrix} = \det(A - BC).$$

Rješenje.

$$\begin{vmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{21} \\ c_{11} & c_{12} & 1 & 0 \\ c_{21} & c_{22} & 0 & 1 \end{vmatrix} \underset{\substack{\text{treći redak pomnožen s } -b_{11} \text{ i } -b_{12} \\ \text{dodajemo u prvi i drugi, redom}}}{=}$$

$$= \begin{vmatrix} a_{11} - b_{11}c_{11} & a_{12} - b_{11}c_{12} & 0 & b_{12} \\ a_{21} - b_{21}c_{11} & a_{22} - b_{21}c_{12} & 0 & b_{21} \\ c_{11} & c_{12} & 1 & 0 \\ c_{21} & c_{22} & 0 & 1 \end{vmatrix}$$

$$\underset{\substack{\text{četvrti redak pomnožen s } -b_{12} \text{ i } -b_{22} \\ \text{dodajemo u prvi i drugi, redom}}}{=}$$

$$= \begin{vmatrix} a_{11} - b_{11}c_{11} - b_{12}c_{21} & a_{12} - b_{11}c_{12} - b_{12}c_{22} & 0 & 0 \\ a_{21} - b_{21}c_{11} - b_{22}c_{11} & a_{22} - b_{21}c_{12} - b_{22}c_{22} & 0 & 0 \\ c_{11} & c_{12} & 1 & 0 \\ c_{21} & c_{22} & 0 & 1 \end{vmatrix}$$

$$\underset{\text{razvoj po zadnjem stupcu}}{=}$$

$$= \begin{vmatrix} a_{11} - b_{11}c_{11} - b_{12}c_{21} & a_{12} - b_{11}c_{12} - b_{12}c_{22} & 0 \\ a_{21} - b_{21}c_{11} - b_{22}c_{11} & a_{22} - b_{21}c_{12} - b_{22}c_{22} & 0 \\ c_{11} & c_{12} & 1 \end{vmatrix}$$

$$\underset{\text{razvoj po zadnjem stupcu}}{=}$$

$$= \begin{vmatrix} a_{11} - b_{11}c_{11} - b_{12}c_{21} & a_{12} - b_{11}c_{12} - b_{12}c_{22} \\ a_{21} - b_{21}c_{11} - b_{22}c_{11} & a_{22} - b_{21}c_{12} - b_{22}c_{22} \end{vmatrix}$$

$$= \det\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} b_{11}c_{11} + b_{12}c_{21} & b_{11}c_{12} + b_{12}c_{22} \\ b_{21}c_{11} + b_{22}c_{11} & b_{21}c_{12} + b_{22}c_{22} \end{pmatrix}\right)$$

$$= \det(A - BC).$$

4. Za $n \in \mathbb{N}$ izračunajte determinantu

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ -1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 0 & -1 & 3 & \dots & n-2 & n-1 & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-2 & n-1 & n \\ 0 & 0 & 0 & \dots & -1 & n-1 & n \\ 0 & 0 & 0 & \dots & 0 & -1 & n \end{vmatrix}.$$

Rješenje.

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ -1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 0 & -1 & 3 & \dots & n-2 & n-1 & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-2 & n-1 & n \\ 0 & 0 & 0 & \dots & -1 & n-1 & n \\ 0 & 0 & 0 & \dots & 0 & -1 & n \end{vmatrix}$$

$$= \text{za sve } 1 \leq i \leq n-1 \text{ tim redom od } i\text{-tog retka oduzmemo } (i+1)\text{. redak}$$

$$= \begin{vmatrix} 2 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 3 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n-1 & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & n & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & n \end{vmatrix}$$

$$= \text{gornjetrokutasta matrica}$$

$$= 2 \cdot 3 \cdots (n-1) \cdot n \cdot n$$

$$= n \cdot n!.$$

5. Izračunajte (ako postoji) inverz matrice

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 9 \\ 1 & 4 & 8 & 12 & 16 \\ 1 & 3 & 7 & 12 & 17 \\ 1 & 3 & 6 & 10 & 15 \end{pmatrix} \in M_5(\mathbb{R}).$$

Rješenje. Ispada

$$A^{-1} = \begin{pmatrix} 4 & -4 & 1 & 0 & 0 \\ -5 & 8 & -4 & 1 & 0 \\ 6 & -10 & 6 & -3 & 1 \\ -4 & 7 & -4 & 3 & -2 \\ 1 & -2 & 1 & -1 & 1 \end{pmatrix}.$$