Linearna algebra 1

2. kolokvij

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1. Za $n \in \mathbb{N}$ izračunajte determinantu

$$D_n = \begin{vmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 1 & 1 & 2 & \dots & n-3 & n-2 & n-1 \\ 1 & 1 & 1 & \dots & n-4 & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 2 & 3 \\ 1 & 1 & 1 & \dots & 1 & 1 & 2 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{vmatrix}.$$

Rješenje.

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 1 & 1 & 2 & \dots & n-3 & n-2 & n-1 \\ 1 & 1 & 1 & \dots & n-4 & n-3 & n-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 2 & 3 \\ 1 & 1 & 1 & \dots & 1 & 1 & 2 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{vmatrix} &= \begin{array}{c}\text{od } i\text{-tog retka oduzmemo } (i+1)\text{-vi redak,} \\ \text{redom za } 1 \le i \le n-1\end{array} \\ &= \begin{vmatrix} 0 & 1 & 2 & \dots & n-3 & n-2 & n-1 \\ 0 & 0 & 1 & \dots & n-4 & n-3 & n-2 \\ 0 & 0 & 0 & \dots & n-5 & n-4 & n-3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 2 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{vmatrix} \\ &= \text{Laplaceov razvoj po prvom retku} \\ &= (-1)^{n+1} \begin{vmatrix} 1 & 2 & \dots & n-3 & n-2 & n-1 \\ 0 & 1 & \dots & n-4 & n-3 & n-2 \\ 0 & 0 & \dots & n-5 & n-4 & n-3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 2 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{vmatrix} \\ &= \text{gornjetrokutasta matrica} \\ &= (-1)^{n+1}. \end{aligned}$$

2. Za koje $\lambda \in \mathbb{R}$ je matrica

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -2 & -1 & 4 & 1 \\ 0 & 0 & \lambda & 0 \\ -2 & 0 & 2 & 1 \end{bmatrix}$$

regularna? Za sve takve λ nađite joj inverz A^{-1}.

Rješenje. Pokušajmo naći inverz matrice A elementarnim transformacijama redaka. Imamo

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & 4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right].$$

Ako je $\lambda = 0$, tada je treći nulredak pa je matrica A ekvivalentna matrici ranga manjeg od 4. Slijedi da je tada A singularna. U nastavku stoga pretpostavljamo da je $\lambda \neq 0$ i dijelimo treći redak tim brojem.

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & \frac{1}{\lambda} & 0 \\ 0 & 1 & 0 & -1 & -2 & -1 & \frac{2}{\lambda} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & \frac{1}{\lambda} & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & \frac{2}{\lambda} & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right].$$

Vidimo da inverz postoji za sve $\lambda \in \mathbb{R} \setminus \{0\}$ i dan je s

$$A^{-1} = \begin{bmatrix} 1 & 0 & \frac{1}{\lambda} & 0 \\ 0 & -1 & \frac{2}{\lambda} & 1 \\ 0 & 0 & \frac{1}{\lambda} & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

3. Za proizvoljnu matricu

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \in M_3(\mathbb{R})$$

dokažite da za sve $x \in \mathbb{R}$ vrijedi formula

$$\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} = (1-x^2)\det A.$$

Rješenje. Koristimo aditivnost determinante s obzirom na stupce:

$$\begin{aligned}
\begin{vmatrix} a_1+b_1x & a_1x+b_1 & c_1 \\ a_2+b_2x & a_2x+b_2 & c_2 \\ a_3+b_3x & a_3x+b_3 & c_3 \end{vmatrix} &= \begin{vmatrix} a_1 & a_1x+b_1 & c_1 \\ a_2 & a_2x+b_2 & c_2 \\ a_3 & a_3x+b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1x+b_1 & c_1 \\ b_2x & a_2x+b_2 & c_2 \\ b_3x & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \det A + x \begin{vmatrix} b_1 & a_1x+b_1 & c_1 \\ b_2 & a_2x+b_2 & c_2 \\ b_3 & a_3x+b_3 & c_3 \end{vmatrix} \\
&= \text{od drugog stupca oduzimamo prvi} \\
&= \det A + x \begin{vmatrix} b_1 & a_1x & c_1 \\ b_2 & a_2x & c_2 \\ b_3 & a_3x & c_3 \end{vmatrix} \\
&= \det A + x^2 \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \\
&= \text{zamjena prva dva stupca} \\
&= \det A - x^2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\
&= (1-x^2)\det A.
\end{aligned}$$

4. Riješite sustav u ovisnosti o parametru $\lambda \in \mathbb{R}$:

$$\begin{cases} x_1 + x_3 + x_4 = \lambda+1, \\ -x_1 - \lambda x_2 - (\lambda+1)x_4 = -\lambda-2, \\ (-\lambda+1)x_2 + 2x_3 - \lambda x_4 = 0, \\ x_1 + x_2 + 2x_3 + (\lambda+2)x_4 = \lambda+2. \end{cases}$$

Rješenje. Kad sustav zapišemo matrično, radimo elementarne transformacije redaka:

$$\begin{aligned}
\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \lambda+1 \\ -1 & -\lambda & 0 & -(\lambda+1) & -\lambda-2 \\ 0 & -\lambda+1 & 2 & -\lambda & 0 \\ 1 & 1 & 2 & \lambda+2 & \lambda+2 \end{array}\right] &\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \lambda+1 \\ 0 & -\lambda & 1 & -\lambda & -1 \\ 0 & -\lambda+1 & 2 & -\lambda & 0 \\ 0 & 1 & 1 & \lambda+1 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \lambda+1 \\ 0 & 0 & \lambda+1 & \lambda^2 & \lambda-1 \\ 0 & 0 & \lambda+1 & \lambda^2-\lambda-1 & \lambda-1 \\ 0 & 1 & 1 & \lambda+1 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \lambda+1 \\ 0 & 1 & 1 & \lambda+1 & 1 \\ 0 & 0 & \lambda+1 & \lambda^2 & \lambda-1 \\ 0 & 0 & \lambda+1 & \lambda^2-\lambda-1 & \lambda-1 \end{array}\right].
\end{aligned}$$

Ako je $\lambda = -1$, tada ova matrica glasi

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

što je ekvivalentno sustavu

$$\begin{cases} x_1 \qquad + x_3 + x_4 = 0, \\ \qquad x_2 + x_3 \qquad = 1, \\ \qquad\qquad\quad x_4 = -2, \\ \qquad\qquad\quad x_4 = -2. \end{cases}$$

pa očitavamo rješenje

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \qquad \alpha \in \mathbb{R}.$$

Nastavljamo dalje uz pretpostavku $\lambda \neq -1$. Tada treći i četvrti redak možemo podijeliti s $\lambda + 1$ pa imamo

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \lambda + 1 \\ 0 & 1 & 1 & \lambda + 1 & 1 \\ 0 & 0 & 1 & \frac{\lambda^2}{\lambda+1} & \frac{\lambda-1}{\lambda+1} \\ 0 & 0 & 1 & \frac{\lambda^2-\lambda-1}{\lambda+1} & \frac{\lambda-1}{\lambda+1} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{-\lambda^2+\lambda+1}{\lambda+1} & \frac{\lambda^2+\lambda+2}{\lambda+1} \\ 0 & 1 & 0 & \frac{2\lambda+1}{\lambda+1} & \frac{2}{\lambda+1} \\ 0 & 0 & 1 & \frac{\lambda^2}{\lambda+1} & \frac{\lambda-1}{\lambda+1} \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{\lambda^2+\lambda+2}{\lambda+1} \\ 0 & 1 & 0 & 0 & \frac{2}{\lambda+1} \\ 0 & 0 & 1 & 0 & \frac{\lambda-1}{\lambda+1} \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Odavde očitavamo jedinstveno rješenje

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{\lambda^2+\lambda+2}{\lambda+1} \\ \frac{2}{\lambda+1} \\ \frac{\lambda-1}{\lambda+1} \\ 0 \end{bmatrix}.$$