Linearna algebra 2

Probni 2. kolokvij

02. 06. 2025.

1. Zadan je linearan operator $A : \mathbb{R}^3 \to \mathbb{R}^3$ svojim prikazom u kanonskoj bazi

$$A(e) = \begin{bmatrix} -2 & 5 & 7 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Nađite bazu (f) za $\mathbb{R}^3$ takvu da je A dijagonalna matrica. Izračunajte $(A^{30})(e)$.

Rješenje. Računamo karakteristični polinom od A:

$$k_A(\lambda) = k_{A(e)}(\lambda) = \det(A(e) - \lambda I) = \begin{vmatrix} -2-\lambda & 5 & 7 \\ 1 & -\lambda & -1 \\ -1 & 1 & 2-\lambda \end{vmatrix} = -\lambda^3 + \lambda = -\lambda(\lambda-1)(\lambda+1).$$

Slijedi da je $\sigma(A) = \{-1, 1, 0\}$. Računamo svojstvene potprostore:

$$V_A(-1) = \ker(A+I) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} -1 & 5 & 7 \\ 1 & 1 & -1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\} = \left\{ t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\},$$

$$V_A(1) = \ker(A-I) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} -3 & 5 & 7 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\} = \left\{ t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} : t \in \mathbb{R} \right\},$$

$$V_A(0) = \ker(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} -2 & 5 & 7 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\} = \left\{ t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Bazu (f) formiramo od svojstvenih vektora:

$$f_1 := \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \qquad f_2 := \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \qquad f_3 := \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Tada znamo da je

$$A(f) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Sada imamo

$$(A^{30})(f) = (A(f))^{30} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{30} = \begin{bmatrix} (-1)^{30} & 0 & 0 \\ 0 & 1^{30} & 0 \\ 0 & 0 & 0^{30} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

U kanonskoj bazi dobivamo

$$(A^{30})(e) = I(e,f)(A^{30})(f)I(f,e) = \begin{bmatrix} -2 & 5 & 7 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & 5 & 7 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}.$$

2. Dokažite da linearan operator

$$A : M_2(\mathbb{R}) \to M_2(\mathbb{R}), \qquad A\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

nije moguće dijagonalizirati.

Rješenje. Označimo s $(e) = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ kanonsku bazu za M_2. Lako izračunamo

$$A(e) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Računamo karakteristični polinom od A:

$$k_A(\lambda) = k_{A(e)}(\lambda) = \det(A(e) - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)^4 = \lambda^4.$$

Slijedi da je $\sigma(A) = \{0\}$. Računamo svojstveni potprostor:

$$V_A(0) = \ker(A) = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

Slijedi da je $\{E_{11}, E_{12}\}$ baza za $V_A(0)$ pa je $g(2) = \dim V_A(0) = 2$. Budući da je suma svih geometrijskih kratnosti operatora A jednaka $g(2) = 2$, što je manje od dimenzije prostora $\dim M_2 = 4$, zaključujemo da operator A nije moguće dijagonalizirati.

3. Na vektorskom prostoru $\mathbb{R}^3$ zadan je skalarni produkt

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle := x_1 y_1 + x_2 y_2 + 2 x_3 y_3.$$

S obzirom na $\langle \cdot, \cdot \rangle$ ortonormirajte skup

$$\{(2, 2, 2), (3, 3, 9), (1, 0, 0)\}.$$

Rješenje. Očito za sve $(x_1, x_2, x_3) \in \mathbb{R}^3$ vrijedi

$$\|(x_1, x_2, x_3)\| = \sqrt{\langle (x_1, x_2, x_3), (x_1, x_2, x_3) \rangle} = \sqrt{x_1^2 + x_2^2 + 2x_3^2}.$$

Imamo

$$\|(2, 2, 2)\| = \sqrt{2^2 + 2^2 + 2 \cdot 2^2} = 4$$

pa je

$$e_1 = \frac{1}{4}(2, 2, 2) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

Imamo

$$\left\langle (3, 3, 9), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \right\rangle = 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} + 2 \cdot 9 \cdot \frac{1}{2} = 12$$

pa je

$$b_2 := (3, 3, 9) - \left\langle (3, 3, 9), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \right\rangle \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = (3, 3, 9) - 12 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) (-3, -3, 3).$$

Imamo

$$\|(-3,-3,3)\| = \sqrt{(-3)^2+(-3)^2+2\cdot 3^2} = 6$$

pa je

$$e_2 := \frac{b_2}{\|b_2\|} = \frac{1}{6}(-3,-3,3) = \left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right).$$

Nadalje, imamo

$$\left\langle (1,0,0), \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right\rangle = \frac{1}{2}, \qquad \left\langle (1,0,0), \left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)\right\rangle = -\frac{1}{2}$$

pa je

$$\begin{aligned} b_3 &:= (1,0,0) - \left\langle (1,0,0), \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right\rangle \left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) - \left\langle (1,0,0), \left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)\right\rangle \left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right) \\ &= (1,0,0) - \frac{1}{2}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) + \frac{1}{2}\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right) \\ &= \left(\frac{1}{2},-\frac{1}{2},0\right). \end{aligned}$$

Napokon, imamo

$$\left\|\left(\frac{1}{2},-\frac{1}{2},0\right)\right\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 2\cdot 0^2} = \frac{1}{\sqrt{2}}$$

pa je

$$e_3 := \frac{b_3}{\|b_3\|} = \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right).$$

Dakle,

$$\{e_1,e_2,e_3\} = \left\{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), \left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right), \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)\right\}$$

je traženi ortonormirani skup.

4. Na unitarnom prostoru $\mathbb{R}^4$ sa standardnim skalarnim produktom zadan je potprostor

$$M := \{(x,x,y,y) : x,y \in \mathbb{R}\}.$$

Odredite $M^\perp$ te neku njegovu ortonormiranu bazu i dimenziju.

Rješenje. Očito je $\{(1,1,0,0),(0,0,1,1)\}$ baza za M. Imamo

$$\begin{aligned} (x_1,x_2,x_3,x_4) \in M^\perp &\iff (x_1,x_2,x_3,x_4) \perp (1,1,0,0),(0,0,1,1) \\ &\iff \begin{cases} \langle (x_1,x_2,x_3,x_4),(1,1,0,0)\rangle = 0, \\ \langle (x_1,x_2,x_3,x_4),(0,0,1,1)\rangle = 0, \end{cases} \\ &\iff \begin{cases} x_1+x_2 = 0, \\ x_3+x_4 = 0, \end{cases} \end{aligned}$$

pa je

$$M^\perp = \{(a,-a,b,-b) : a,b \in \mathbb{R}\}.$$

Vidimo da je $\{(1,-1,0,0),(0,0,1,-1)\}$ baza za $M^\perp$ pa je $\dim M^\perp = 2$. Očito su vektori već ortogonalni pa ih je dovoljno normirati da bismo došli do ortonormirane baze za $M^\perp$. Imamo

$$\|(1,-1,0,0)\| = \|(0,0,1,-1)\| = \sqrt{2}$$

pa je

$$\left\{\frac{1}{\sqrt{2}}(1,-1,0,0), \frac{1}{\sqrt{2}}(1,-1,0,0)\right\}$$

ONB za $M^\perp$.